arXiv:quant-ph/0504060v1 8 Apr 2005

On relations between stable and Zeno dynamics in a leaky graph decay model

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Abstract. We use a caricature model of a system consisting of a quantum wire and a finite number of quantum dots, to discuss relation between the Zeno dynamics and the stable one which governs time evolution of the dot states in the absence of the wire. We analyze the weak coupling case and argue that the two time evolutions can differ significantly only at times comparable with the lifetime of the unstable system undisturbed by perpetual measurement.

1. Introduction

It is well known that the decay of an unstable system can be slowed down, or even fully stopped in the ideal case, if one checks frequently whether the system is still undecayed. The first proper statement of this fact is due to Beskow and Nilsson [2] and a rigorous mathematical proof was given by Friedman [11], but it became popular only after Misra and Sudarshan [14] invented the name "quantum Zeno effect" for it. In recent years this subject attracted a new wave of interest – a rich bibliography can be found, e.g., in [10, 15].

The motivation of this interest is twofold. On one hand the progress in experimental methods makes real the possibility to observe the effect as a phenomenon really existing in the nature, and ultimately to make use of it. On the other hand, the problem presents also interesting mathematical challenges. The most important among them is obviously the question about the *quantum Zeno dynamics:* if the perpetual measurement keeps the state of the system within the Hilbert space associated with the unstable system, what is then the time evolution of such a state? Some recent results [7, 8] give partial answers to this question, which we shall describe below, and there are counterexamples [13], see also [6, Rem. 2.4.9], which point out the borders beyond which it has no sense.

In this note we are going to address a different question. Suppose that at the beginning the interaction responsible for the decay is absent, so state vectors evolve within the mentioned space which we below call $P\mathcal{H}$. Switching the interaction

Received by the editors February 16, 2005.
1991 *Mathematics Subject Classification.* Primary 81V99; Secondary 47D08, 35J10.
Key words and phrases. Zeno dynamics, Schrödinger operator, singular interactions.

with the "environment" in, we allow the system to decay which means the state vectors may partially or fully leave the space $P\mathcal{H}$. If we now perform the Zeno-style monitoring, the system is forced to stay within $P\mathcal{H}$ and to evolve there, but what is in this case the relation of its dynamics to the original "stable" one?

A general answer to this question is by no means easy and we do not strive for this ambitious goal here. Our aim is to analyze a simple example which involves a Schrödinger operator in $L^2(\mathbb{R}^2)$ with a singular interaction supported by a line and a finite family of points [9]. This model is explicitly solvable and can be regarded as a caricature description of a system consisting of a quantum wire and dots which are not connected mutually but can interact by means tunneling. The main result of this paper given in Theorem 6.1 below is that in the model *the two dynamics do not differ significantly during time periods short at the scale given by the lifetime of the system unperturbed by the perpetual observation.*

Let us briefly summarize the contents of the paper. First we recall basic notions concerning Zeno dynamics; we will prove the needed existence result in case when the state spece of the unstable system has a finite dimension. Sections 3–5 are devoted to the mentioned solvable model. We will introduce its Hamiltonian and find its resolvent. Then we will show that in the "weak-coupling" case when the points are sufficiently far from the line the model exhibit resonances, and in Sec. 5 we will treat the model from the decay point of view, showing how the point-interaction eigenfunctions dissipate due to the tunneling between the points and the line; in the appendix we demonstrate that in the weak-coupling case the decay is approximately exponential. The main result is stated and proved in Sec. 6.

2. Quantum Zeno dynamics

Following general principles of quantum decay kinematics [6, Chap. 1] we associate with an unstable system three objects: the state Hilbert space $\mathcal{H}$ describing all of its states including the decayed ones, the full Hamiltonian H on $\mathcal{H}$ and the projection P which specifies the subspace of states of the unstable system alone. H is, of course, a self-adjoint operator, we need to assume that it is *bounded from below.*

The question about the existence of Zeno dynamics mentioned above can be then stated in this context generally as follows: does the limit

$$(P\mathrm{e}^{-iHt/n}P)^n \longrightarrow \mathrm{e}^{-iH_P t} \tag{2.1}$$

hold as $n \to \infty$, in which sense, and what is in such a case the operator H_P? Let us start from the end and consider the quadratic form $u \mapsto \|H^{1/2}Pu\|^2$ with the form domain $D(H^{1/2}P)$ which is closed but in general it may not be densely defined. The classical results of Chernoff [3, 4] suggest that the operator associated with this form, $H_P := (H^{1/2}P)^*(H^{1/2}P)$, is a natural candidate for the generator of the Zeno dynamics, and the counterexamples mentioned in the introduction show that the limit may not exist if H_P is not *densely defined,* so we adopt this assumption.

Remark 2.1. Notice that the operator H_P is an extension of PHP, but in general a nontrivial one. This can be illustrated even in the simplest situation when $\dim P = 1$, because if H is unbounded $D(H)$ is a proper subspace of $D(H^{1/2})$. Take $\psi_0 \in D(H^{1/2})\setminus D(H)$ such that $H^{1/2}\psi_0$ is nonzero, and let P refer to the one-dimensional subspace spanned by ψ_0. This means that PHP cannot be applied to any nonzero vector ψ $(= \alpha\psi_0)$ of $P\mathcal{H}$ while $H_P\psi$ is well defined and nonzero.

It is conjectured that formula (2.1) will hold under the stated assumptions in the strong operator topology. Proof of this claim remains an open question, though. The best result to the date [7] establishes the convergence in a weaker topology which includes averaging of the norm difference with respect to the time variable. While this may be sufficient from the viewpoint of physical interpretation, mathematically the situation is unsatisfactory, since other results available to the date require modifications at the left-hand side of (2.1), either by replacement of the exponential by another Kato function, or by adding a spectral projection interpreted as an additional energy measurement – see [8] for more details.

There is one case, however, when the formula can be proven, namely the situation when $\dim P < \infty$ and the density assumption simply means that $P\mathcal{H} \subset \mathcal{Q}(H)$, where $\mathcal{Q}(H)$ is the form domain of H. Since this exactly what we need for our example, let us state the result.

Theorem 2.2. *Let H be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$, bounded from below, and P a finite-dimensional orthogonal projection on $\mathcal{H}$. If $P\mathcal{H} \subset \mathcal{Q}(H)$, then for any $\psi \in \mathcal{H}$ and $t \geq 0$ we have*

$$\lim_{n\to\infty} (P\mathrm{e}^{-iHt/n}P)^n\psi = \mathrm{e}^{-iH_Pt}\psi\,, \tag{2.2}$$

uniformly on any compact interval of the time variable t.

Proof. The claim can be proved in different ways, see [7] and [8]. Here we use another argument the idea of which was suggested by G.M. Graf and A. Guekos [12]. Notice first that without loss of generality we may suppose that H is strictly positive, i.e. $H \geq \delta I$ for some positive number δ. The said argument is then based on the observation that

$$\lim_{t\to 0} t^{-1}\left\| Pe^{-iHt}P - P\mathrm{e}^{-itH_Pt}P\right\| = 0 \tag{2.3}$$

implies $\left\|(Pe^{-iHt/n}P)^n - Pe^{-iH_Pt}\right\| = n\,o(t/n)$ as $n\to\infty$ by means of a natural telescopic estimate. To establish (2.3) one has first to check that

$$t^{-1}\left[(\phi, Pe^{-iHt}P\psi) - (\phi,\psi) - it(H^{1/2}P\phi, H^{1/2}P\psi)\right] \to 0$$

as $t\to 0$ for all ϕ,ψ from $D(H^{1/2}P)$ which coincides in this case with $P\mathcal{H}\oplus(I-P)\mathcal{H}$ by the closed-graph theorem. The last expression is equal to

$$\left(H^{1/2}P\phi, \left[\frac{e^{-iHt}-I}{Ht} - i\right]H^{1/2}P\psi\right)$$

and the square bracket tends to zero strongly by the functional calculus, which yields the sought conclusion. In the same way we find that

$$t^{-1}\left[(\phi, Pe^{-iH_P t}P\psi) - (\phi,\psi) - it(H_P^{1/2}\phi, H_P^{1/2}\psi)\right] \to 0$$

holds as $t \to 0$ for any vectors $\phi, \psi \in P\mathcal{H}$. Next we note that $(H_P^{1/2}\phi, H_P^{1/2}\psi) = (H^{1/2}P\phi, H^{1/2}P\psi)$ by definition, and consequently, the expression contained in (2.3) tends to zero weakly as $t \to 0$, however, in a finite dimensional $P\mathcal{H}$ the weak and operator-norm topologies are equivalent. $\square$

Remark 2.3. It is clear that the finite dimension of P is essential for the proof. The same results holds for the backward time evolution, $t \le 0$. Moreover, the formula (2.2) has non-symmetric versions with the operator product replaced with $(P\mathrm{e}^{-iHt/n})^n$ and $(\mathrm{e}^{-iHt/n}P)^n$ tending to the same limit – see [7].

3. A model of leaky line and dots

Before coming to the proper decay problem let us describe the general setting of the model. We will consider a generalized Schrödinger operator in $L^2 \equiv L^2(\mathbb{R}^2)$ with a singular interaction supported by a set consisting of two parts. One is a straight line, the other is a finite family of points situated in general outside the line, hence formally we can write our Hamiltonian as

$$-\Delta - \alpha\delta(x-\Sigma) + \sum_{i=1}^{n} \tilde\beta_i \delta(x - y^{(i)})\,, \tag{3.1}$$

where $\alpha > 0$, $\Sigma := \{(x_1, 0);\, x_1 \in \mathbb{R}^2\}$, and $\Pi := \{y^{(i)}\}_{i=1}^n \subset \mathbb{R}^2 \setminus \Sigma$. The formal coupling constants of the two-dimensional δ potentials are marked by tildes because they are not identical with the proper coupling parameters β_i which define these point interaction by means of appropriate boundary conditions.

Following the standard prescription [1] one can define the operator rigorously [9] by introducing appropriated boundary conditions on $\Sigma\cup\Pi$. Consider functions $\psi \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus (\Sigma\cup\Pi)) \cap L^2$ which are continuous on Σ. For a small enough $\rho > 0$ the restriction $\psi\restriction_{\mathcal{C}_{\rho,i}}$ to the circle $\mathcal{C}_{\rho,i} \equiv \mathcal{C}_\rho(y_i) := \{q \in \mathbb{R}^2 : |q - y^{(i)}| = \rho\}$ is well defined; we will say that ψ belongs to $D(\dot H_{\alpha,\beta})$ *iff* $(\partial^2_{x_1} + \partial^2_{x_2})\psi$ on $\mathbb{R}^2 \setminus (\Sigma\cup\Pi)$ in the sense of distributions belongs to L^2 and the limits

$$\begin{aligned}
&\Xi_i(\psi) := -\lim_{\rho\to 0}\frac{1}{\ln\rho}\,\psi\restriction_{\mathcal{C}_{\rho,i}}\,,\ \Omega_i(\psi) := \lim_{\rho\to 0}[\psi\restriction_{\mathcal{C}_{\rho,i}} + \Xi_i(\psi)\ln\rho]\,,\ \ i = 1,\dots,n\,,\\
&\Xi_\Sigma(\psi)(x_1) := \partial_{x_2}\psi(x_1, 0+) - \partial_{x_2}\psi(x_1, 0-)\,,\quad \Omega_\Sigma(\psi)(x_1) := \psi(x_1, 0)
\end{aligned}$$

exist, they are finite, and satisfy the relations

$$2\pi\beta_i\Xi_i(\psi) = \Omega_i(\psi)\,,\quad \Xi_\Sigma(\psi)(x_1) = -\alpha\Omega_\Sigma(\psi)(x_1)\,, \tag{3.2}$$

where $\beta_i \in \mathbb{R}$ are the true coupling parameters; we put $\beta \equiv (\beta_1, \dots, \beta_n)$ in the following. On this domain we define the operator $\dot{H}_{\alpha,\beta} : D(\dot{H}_{\alpha,\beta}) \to L^2$ by

$$\dot{H}_{\alpha,\beta}\psi(x) = -\Delta\psi(x) \quad \text{for} \quad x \in \mathbb{R}^2 \setminus (\Sigma \cup \Pi) .$$

It is now a standard thing to check that $\dot{H}_{\alpha,\beta}$ is essentially self-adjoint [9]; we identify its closure denoted as $H_{\alpha,\beta}$ with the formal Hamiltonian (3.1).

To find the resolvent of $H_{\alpha,\beta}$ we start from $R(z) = (-\Delta - z)^{-1}$ which is for any $z \in \mathbb{C} \setminus [0,\infty)$ an integral operator with the kernel $G_z(x,x') = \frac{1}{2\pi} K_0(\sqrt{-z}|x-x'|)$, where K_0 is the Macdonald function and $z \mapsto \sqrt{z}$ has conventionally a cut at the positive halfline; we denote by $\mathbf{R}(z)$ the unitary operator with the same kernel acting from L^2 to $W^{2,2} \equiv W^{2,2}(\mathbb{R}^2)$. We introduce two auxiliary Hilbert spaces, $\mathcal{H}_0 := L^2(\mathbb{R})$ and $\mathcal{H}_1 := \mathbb{C}^n$, and the corresponding trace maps $\tau_j : W^{2,2} \to \mathcal{H}_j$ which act as

$$\tau_0\psi := \psi\!\upharpoonright_\Sigma , \quad \tau_1\psi := \psi\!\upharpoonright_\Pi = (\psi\!\upharpoonright_{\{y^{(1)}\}}, \dots, \psi\!\upharpoonright_{\{y^{(n)}\}}) ,$$

respectively; they allow us to define the canonical embeddings of $\mathbf{R}(z)$ to $\mathcal{H}_i$, i.e.

$$\mathbf{R}_{iL}(z) = \tau_i R(z) : L^2 \to \mathcal{H}_i , \quad \mathbf{R}_{Li}(z) = [\mathbf{R}_{iL}(z)]^* : \mathcal{H}_i \to L^2 ,$$

and $\mathbf{R}_{ji}(z) = \tau_j \mathbf{R}_{Li}(z) : \mathcal{H}_i \to \mathcal{H}_j$, all expressed naturally through the free Green's function in their kernels, with the variable range corresponding to a given $\mathcal{H}_i$. The operator-valued matrix $\Gamma(z) = [\Gamma_{ij}(z)] : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1$ is defined by

$$\begin{aligned}
\Gamma_{ij}(z)g &:= -\mathbf{R}_{ij}(z)g \qquad \text{for} \quad i \neq j \quad \text{and} \quad g \in \mathcal{H}_j , \\
\Gamma_{00}(z)f &:= \left[\alpha^{-1} - \mathbf{R}_{00}(z)\right] f \qquad \text{if} \quad f \in \mathcal{H}_0 , \\
\Gamma_{11}(z)\varphi &:= \left[s_{\beta_l}(z)\delta_{kl} - G_z(y^{(k)}, y^{(l)})(1-\delta_{kl})\right]_{k,l=1}^{n} \varphi \quad \text{for} \quad \varphi \in \mathcal{H}_1 ,
\end{aligned}$$

where $s_{\beta_l}(z) = \beta_l + s(z) := \beta_l + \frac{1}{2\pi}(\ln \frac{\sqrt{z}}{2i} - \psi(1))$ and $-\psi(1)$ is the Euler number.

For z from $\rho(H_{\alpha,\beta})$ the operator $\Gamma(z)$ is boundedly invertible. In particular, $\Gamma_{00}(z)$ is invertible and it makes sense to define $D(z) \equiv D_{11}(z) : \mathcal{H}_1 \to \mathcal{H}_1$ by

$$D(z) = \Gamma_{11}(z) - \Gamma_{10}(z)\Gamma_{00}(z)^{-1}\Gamma_{01}(z) \tag{3.3}$$

which we call the *reduced determinant* of Γ; it allows us to write the inverse of $\Gamma(z)$ as $[\Gamma(z)]^{-1} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1$ with the "block elements" defined by

$$\begin{aligned}
[\Gamma(z)]_{11}^{-1} &= D(z)^{-1} , \\
[\Gamma(z)]_{00}^{-1} &= \Gamma_{00}(z)^{-1} + \Gamma_{00}(z)^{-1}\Gamma_{01}(z)D(z)^{-1}\Gamma_{10}(z)\Gamma_{00}(z)^{-1} , \\
[\Gamma(z)]_{01}^{-1} &= -\Gamma_{00}(z)^{-1}\Gamma_{01}(z)D(z)^{-1} , \\
[\Gamma(z)]_{10}^{-1} &= -D(z)^{-1}\Gamma_{10}(z)\Gamma_{00}(z)^{-1} ;
\end{aligned}$$

in the above formulae we use notation $\Gamma_{ij}(z)^{-1}$ for the inverse of $\Gamma_{ij}(z)$ and $[\Gamma(z)]_{ij}^{-1}$ for the matrix element of $[\Gamma(z)]^{-1}$.

Before using this to express $R_{\alpha,\beta}(z) \equiv (H_{\alpha,\beta} - z)^{-1}$ we introduce another notation which allow us to write $R_{\alpha,\beta}(z)$ through a perturbation of the "line only"

Hamiltonian $\tilde{H}_\alpha$ the resolvent of which is the integral operator

$$R_\alpha(z) = R(z) + R_{L0}(z)\Gamma_{00}^{-1}R_{0L}(z)$$

for $z \in \mathbb{C} \setminus [-\frac{1}{4}\alpha^2, \infty)$. We define $\mathbf{R}_{\alpha;L1}(z) : \mathcal{H}_1 \to L^2$ and $\mathbf{R}_{\alpha;1L}(z) : L^2 \to \mathcal{H}_1$ by

$$\mathbf{R}_{\alpha;1L}(z)\psi := R_\alpha(z)\psi \restriction_\Pi \quad \text{for } \psi \in L^2$$

and $\mathbf{R}_{\alpha;L1}(z) := \mathbf{R}^*_{\alpha;1L}(z)$; the resolvent difference between $H_{\alpha,\beta}$ and $\tilde{H}_\alpha$ is given then by Krein's formula. Now we can state the result; for the proof and a more detailed discussion we refer to [9].

Theorem 3.1. *For any $z \in \rho(H_{\alpha,\beta})$ with $\operatorname{Im} z > 0$ we have*

$$R_{\alpha,\beta}(z) = R(z) + \sum_{i,j=0}^{1} \mathbf{R}_{Li}(z)[\Gamma(z)]_{ij}^{-1}\mathbf{R}_{jL}(z) = R_\alpha(z) + \mathbf{R}_{\alpha;L1}(z)D(z)^{-1}\mathbf{R}_{\alpha;1L}(z)\,.$$

These formulæ make it possible to analyze spectral properties of the operator $H_{\alpha,\beta}$, see again [9] for more details. In this paper we will be concerned with one aspect of this problem only, namely with perturbations of embedded eigenvalues.

4. Resonance poles

The decay in our model is due to the tunneling between the points and the line. This interaction is "switched off" if the line is removed (formally speaking, put to an infinite distance). Consequently, the free Hamiltonian from the decay point of view is the point interaction only $\tilde{H}_\beta := H_{0,\beta}$. Depending on the configuration of the set Π and the coupling parameters β this operator has m eigenvalues, $1 \le m \le n$. *We will always assume* in the following that they satisfy the condition

$$-\frac{1}{4}\alpha^2 < \epsilon_1 < \cdots < \epsilon_m < 0 \quad \text{and} \quad m > 1\,, \tag{4.1}$$

i.e., the discrete spectrum of $\tilde{H}_\beta$ is simple, contained in (the negative part of) $\sigma(\tilde{H}_\alpha) = \sigma_{ac}(H_{\alpha,\beta}) = (-\alpha^2/4, \infty)$, and consists of more than a single point. Let us specify the interactions sites by their Cartesian coordinates, $y^{(i)} = (c_i, a_i)$. We also introduce the notations $a = (a_1, ..., a_n)$ and $d_{ij} = |y^{(i)} - y^{(j)}|$ for the distances between point interactions.

To find resonances in our model we will rely on a Birman-Schwinger type argument [1]. More specifically, our aim is to find poles of the resolvent through zeros of the operator-valued function (3.3). First we have to find a more explicit form of $D(\cdot)$; having in mind that resonance poles have to be looked for on the second sheet we will derive the analytical continuation of $D(\cdot)$ to a subset Ω_- of the lower halfplane across the segment $(-\alpha^2/4, 0)$ of the real axis; for the sake of definiteness we employ the notation $D(\cdot)^{(l)}$ where $l = -1, 0, 1$ refers to the argument z from Ω_-, the segment $(-\alpha^2/4, 0)$, and the upper halfplane, $\operatorname{Im} z > 0$,

[1] We will follow here the idea which was precisely discussed in [9]

respectively. Using the resolvent formula of the previous section we see that the first component of the operator $\Gamma_{11}(\cdot)^{(l)}$ is the $n\times n$ matrix with the elements

$$\Gamma_{11;jk}(\cdot)^{(l)} = -\frac{1}{2\pi}K_0(d_{jk}\sqrt{-\cdot}) \quad \text{for } j\neq k$$

and

$$\Gamma_{11;jj}(\cdot)^{(l)} = \beta_j + 1/2\pi(\ln\sqrt{(-\cdot)} - \psi(1))$$

for every l. To find an explicit form of the second component let us introduce

$$\mu_{ij}(z,t) := \frac{i\alpha}{2^5\pi}\frac{(\alpha-2i(z-t)^{1/2})\,\mathrm{e}^{i(z-t)^{1/2}(|a_i|+|a_j|)}}{t^{1/2}(z-t)^{1/2}}\,\mathrm{e}^{it^{1/2}(c_i-c_j)}$$

and $\mu^0_{ij}(\lambda,t) := \lim_{\eta\to0+}\mu_{ij}(\lambda+i\eta,t)$ cf. [9]. Using this notation we can rewrite the matrix elements of $(\Gamma_{10}\Gamma_{00}^{-1}\Gamma_{01})^{(\cdot)}(\cdot)$ in the following form,

$$\begin{aligned}
\theta^{(0)}_{ij}(\lambda) &= \mathcal{P}\int_0^\infty \frac{\mu^0_{ij}(\lambda,t)}{t-\lambda-\alpha^2/4}\,\mathrm{d}t + g_{\alpha,ij}(\lambda)\,, \qquad \lambda\in(-\frac{\alpha^2}{4},0)\\
\theta^{(l)}_{ij}(z) &= l\int_0^\infty \frac{\mu_{ij}(z,t)}{t-z-\alpha^2/4}\,\mathrm{d}t + (l-1)g_{\alpha,ij}(z) \quad \text{for } \; l=1,\,-1
\end{aligned}$$

where $\mathcal{P}$ means the principal value and

$$g_{\alpha,ij}(z) := \frac{i\alpha}{(z+\alpha^2/4)^{1/2}}\,\mathrm{e}^{-\alpha(|a_i|+|a_j|)/2}\,\mathrm{e}^{i(z+\alpha^2/4)^{1/2}(c_i-c_j)}\,.$$

Proceeding in analogy with [9] we evaluate the determinant of $D(\cdot)^{(\cdot)}$ as

$$d(z)^{(l)} \equiv d(a,z)^{(l)} = \sum_{\pi\in\mathcal{P}_n}\operatorname{sgn}\pi\left(\sum_{j=1}^n(-1)^j(S^j_{p_1,\dots,p_n})^{(l)} + \Gamma_{11;1p_1}\dots\Gamma_{11;np_n}\right)(z)\,,$$

where $\mathcal{P}_n$ denotes the permutation group of n elements, $\pi=(p_1,\dots,p_n)$, and

$$(S^j_{p_1,\dots,p_n})^{(l)} = \theta^{(l)}_{jp_1}A^j_{p_2,\dots,p_n}$$

with

$$A^j_{i_2,\dots,i_n} := \begin{cases}\Gamma_{11;1i_2}\dots\Gamma_{11;j-1,i_j}\Gamma_{11;j+1,i_{j+1}}\dots\Gamma_{11;ki_k} & \text{if} \quad j>1\\ \Gamma_{11;2i_2}\dots\Gamma_{11;ki_k} & \text{if} \quad j=1\end{cases}$$

After this preliminary we want to find roots of the equation $d(a,z)^{(l)}(z)=0$. On a heuristic level the resonances are due to tunneling between the line and the points, thus it is convenient to introduce the following reparametrization,

$$\tilde{b}(a)\equiv(b_1(a),\dots,b_n(a)) \quad b_i(a)=\mathrm{e}^{-|a_i|\sqrt{-\epsilon_i}}$$

and to put $\eta(\tilde{b},z) = d^{(-1)}(a,z)$. As we have said the absence of the straight-line interaction can be regarded in a sense as putting the line to an infinite distance from the points, thus corresponding to $\tilde{b}=0$. In this case we have

$$\eta(0,z) = \sum_{\pi\in\mathcal{P}_n}\operatorname{sgn}\pi\,(\Gamma_{11;1p_1}\dots\Gamma_{11;np_n})\,(z) = \det\Gamma_{11}(z)\,,$$

so the roots of the equation $\eta(0,z)=0$ are nothing else than the eigenvalues of the point-interaction Hamiltonian $\tilde{H}_\beta$; with the condition (4.1) in mind we have

$$\eta(0,\epsilon_i)=0\,,\quad i=1,...,m\,.$$

Now one proceeds as in [9] checking that the hypotheses of the implicit-function theorem are satisfied; then the equation $\eta(\tilde{b},z)=0$ has for all the b_i small enough just m zeros which admit the following weak-coupling asymptotic expansion,

$$z_i(b)=\epsilon_i+\mathcal{O}(b)+i\mathcal{O}(b) \quad \text{where} \quad b:=\max_{1\le i\le m} b_i\,. \tag{4.2}$$

Remark 4.1. If $n\ge 2$ there can be eigenvalues of $\tilde{H}_\beta$ which remain embedded under the line perturbation due to a symmetry; the simplest example is a pair of point interactions with the same coupling and mirror symmetry with respect to Σ. From the viewpoint of decay which is important in this paper they represent a trivial case which we exclude in the following. Neither shall we consider resonances which result from a slight violation of such a symmetry – cf. a discussion in [9].

5. Decay of the dot states

As usual the resonance poles discussed above can be manifested in two ways, either in scattering properties, here of a particle moving along the "wire" Σ, or through the time evolution of states associated with the "dots" Π. By assumption (4.1) there is a nontrivial discrete spectrum of $\tilde{H}_\beta$ embedded in $(-\frac{1}{4}\alpha^2,0)$. Let us denote the corresponding normalized eigenfunctions ψ_j , $j=1,\dots,m$, given by

$$\psi_j(x)=\sum_{i=1}^m d_i^{(j)}\phi_i^{(j)}(x)\,,\quad \phi_i^{(j)}(x):=\sqrt{-\frac{\epsilon_j}{\pi}}\,K_0(\sqrt{-\epsilon_j}|x-y^{(i)}|) \tag{5.1}$$

in accordance with [1, Sec. II.3], where the vectors $d^{(j)}\in\mathbb{C}^m$ satisfy the equation

$$\Gamma_{11}(\epsilon_j)d^{(j)}=0 \tag{5.2}$$

and a normalization condition which in view of $\|\phi_i^{(j)}\|=1$ reads

$$|d^{(j)}|^2+2\mathrm{Re}\sum_{i=2}^m\sum_{k=1}^{i-1}\overline{d_i^{(j)}}d_k^{(j)}(\phi_i^{(j)},\phi_k^{(j)})=1\,. \tag{5.3}$$

In particular, if the distances between the points of Π are large (the natural length scale is given by $(-\epsilon_j)^{-1/2}$), the cross terms are small and $|d^{(j)}|$ is close to one.

Let us now *specify the unstable system* of our model by identifying its state Hilbert space $P\mathcal{H}$ with the span of the vectors $\psi_1,\dots,\psi_m$. Suppose that it is prepared at the initial instant $t=0$ at a state $\psi\in P\mathcal{H}$, then the decay law describing the probability of finding the system undecayed at a subsequent measurement performed at t, without disturbing it in between [6], is

$$P_\psi(t)=\|P\mathrm{e}^{-iH_{\alpha,\beta}t}\psi\|^2. \tag{5.4}$$

We are particularly interested in the *weak-coupling situation* where the distance between Σ and Π is a large at the scale given by $(-\epsilon_m)^{-1/2}$. Since our model bears resemblance with the (multidimensional) Friedrichs model one can conjecture in analogy with [5] that the leading term in $P_\psi(t)$ will come from the appropriate semigroup evolution on $P\mathcal{H}$, in particular, for the basis states ψ_j we will have a dominantly exponential decay, $P_{\psi_j}(t) \approx \mathrm{e}^{-\Gamma_j t}$ with $\Gamma_j = 2\,\mathrm{Im}\, z_j(b)$. A precise discussion of this question is postponed to appendix – see Sec. 7 below.

Remark 5.1. The quantities Γ_j^{-1} provide thus a natural time scale for the decay and we will use $\max_j \Gamma_j^{-1}$ as a measure of the system lifetime. A *caveat* is needed, however, with respect to the notion of lifetime [6] which is conventionally defined as $T_\psi = \int_0^\infty P_\psi(t)\,\mathrm{d}t$. It has been shown in [9] that $P\mathcal{H}$ is not contained is the absolutely continuous subspace of $H_{\alpha,\beta}$ if $n=1$, and the argument easily extends to any $n \in \mathbb{N}_0$. This means that a part of the original state survives as $t \to \infty$, even if it is a small one in the weak-coupling case. It is a long-time effect, of course, which has no relevance for the problem considered here.

6. Stable and Zeno dynamics in the model

Suppose now finally that we perform the Zeno time at our decaying system characterized by the operator $H_{\alpha,\beta}$ and the projection P. The latter has by assumption the dimension $1 < m < \infty$ and it is straightforward to check that $P\mathcal{H} \subset \mathcal{Q}(H_{\alpha,\beta})$. Moreover the form associated with generator H_P has in the quantum-dot state basis the following matrix representation

$$(\psi_j, H_P\psi_k) = \delta_{jk}\epsilon_j - \alpha \int_\Sigma \bar{\psi}_j(x_1,0)\psi_k(x_1,0)\,\mathrm{d}x_1\,, \tag{6.1}$$

where the first term corresponds, of course, to the “dots-only” operator $\tilde{H}_\beta$.

Theorem 6.1. *The two dynamics do not differ significantly for times satisfying*

$$t \ll C\,\mathrm{e}^{2\sqrt{-\epsilon}|\tilde{a}|}\,, \tag{6.2}$$

where C is a positive constant and $|\tilde{a}| = \min_i |a_i|$, $\epsilon = \max_i \epsilon_i$.

Proof. The difference is characterized by the operator $\mathcal{U}_t := (\mathrm{e}^{-i\tilde{H}_\beta t} - \mathrm{e}^{-iH_P t})P$. Taking into account the unitarity of its parts together with a functional calculus estimate based on $|\mathrm{e}^{iz}-1| \le |z|$ we find that the norm of $\mathcal{U}_t$ remains small as long as $t\|(\tilde{H}_\beta - H_P)P\| \ll 1$. Thus to check (6.2) we have to estimate norm of the operator $(\tilde{H}_\beta - H_P)P$ acting in $P\mathcal{H}$; in the basis of the vectors $\{\psi_j\}_{j=1}^m$ it is represented by $m\times m$ matrix with the elements

$$s_{ij} = \alpha(\psi_i,\psi_j)_\Sigma\,,$$

where $(\psi_i,\psi_j)_\Sigma := \int_\Sigma \bar{\psi}_i(x_1,0)\psi_j(x_1,0)\,\mathrm{d}x_1$. Using the representation (5.1) we obtain

$$s_{ij} = \alpha \sum_{(l,k)\in M\times M} \bar{d}_l^{(i)} d_k^{(j)} (\phi_l^{(i)}, \phi_k^{(j)})_\Sigma$$

where M is a shorthand for $(1,...,m)$. To proceed further we use Schur-Holmgren bound by which the norm of $(\tilde{H}_\beta - H_P)P$ does not exceed mS, where $S := \max_{(i,j)\in M\times M}|s_{ij}|$, and the last named quantity can be estimated by

$$S \le \alpha m^2 \max_{(i,j,k,l)\in M^4} |\bar{d}_l^{(i)} d_j^{(k)} (\phi_l^{(i)}, \phi_k^{(j)})_\Sigma| \,.$$

The final step is to estimate the expressions $(\phi_l^{(i)}, \phi_k^{(j)})_\Sigma$. Using the momentum representation of Macdonald function we obtain

$$(\phi_l^{(i)}, \phi_k^{(j)})_\Sigma = \frac{\sqrt{\epsilon_i\epsilon_j}}{2}\int_{\mathbb{R}} \frac{\mathrm{e}^{-((p_1^2-\epsilon_i)^{1/2}|a_l|-(p_1^2-\epsilon_j)^{1/2}|a_k|)}}{(p_1^2-\epsilon_i)^{1/2}(p_1^2-\epsilon_j)^{1/2}}\,\mathrm{e}^{ip_1(c_k-c_l)}\,\mathrm{d}p_1\,,$$

where $y^{(i)} = (c_i, a_i)$ as before. A simple estimate of the above integral yields

$$(\phi_l^{(i)}, \phi_k^{(j)})_\Sigma \le \frac{\pi}{2}\,\frac{\epsilon_{min}}{\sqrt{-\epsilon}}\,\mathrm{e}^{-2\sqrt{-\epsilon}|a|}$$

where $\epsilon_{min} = \min_i \epsilon_i$, $|\tilde{a}| = \min_i |a_i|$, and $\epsilon = \max_i \epsilon_i$. In conclusion, we get the bound

$$\|(\tilde{H}_\beta - H_P)P\| \le C\mathrm{e}^{-2\sqrt{-\epsilon}|a|}\,,$$

where $C := \frac{1}{2}\pi m^3 \alpha\, \epsilon_{min}(-\epsilon)^{-1/2} \max_{(i,j,k,l)\in M^4} |\bar{d}_l^{(i)} d_j^{(k)}|$. ☐

7. Appendix: pole approximation for the decaying states

Let us now return to the claim that the decay is approximately exponential when the distances of the points from the line are large. Let ψ_j be the j-th eigenfunction of the point-interaction Hamiltonian $\tilde{H}_\beta$ with the eigenvalue ϵ_j; the related one-dimensional projection will be denoted P_j. Then we make the following claim.

Theorem 7.1. *Suppose that $H_{\alpha,\beta}$ has no embedded eigenvalues. Then in the limit $b \to 0$ where b is defined in (4.2) we have, pointwise in $t \in (0,\infty)$,*

$$\|P_j \mathrm{e}^{-iH_{\alpha,\beta}t}\psi_j - \mathrm{e}^{-iz_jt}\psi_j\| \to 0\,.$$

To prove the theorem we need some preliminaries. For simplicity, we denote $U_t(\epsilon) := \mathrm{e}^{-i\epsilon t}$ for a fixed $t > 0$. It was shown in [9] that the operator $H_{\alpha,\beta}$ has at least one and at most n isolated eigenvalues. We denote them by $\epsilon_{\alpha\beta,k}$, $k = 1,...,l$ with $l \le n$, and use $\psi_{\alpha\beta,k}$ as symbols for the corresponding (normalized) eigenfunctions. Then the spectral theorem gives

$$P_j\,\mathrm{e}^{-iH_{\alpha,\beta}t}\psi_j = \sum_{k=1}^{m} U_t(\epsilon_{\alpha\beta,k})|(\psi_j,\psi_{\alpha\beta,k})|^2\psi_j + P_j\int_{-\alpha^2/4}^{\infty} U_t(\lambda)\mathrm{d}E(\lambda)\psi_j\,, \tag{7.1}$$

where $E(\cdot) \equiv E_{\alpha,\beta}(\cdot)$ is the spectral measure of $H_{\alpha,\beta}$. By assumption there are no embedded eigenvalues (cf. Remark 4.1) and by [9] also the singularly continuous component is void, hence the second term is associated solely with $\sigma_{\mathrm{ac}}(H_{\alpha,\beta})$. Let us first look at this contribution to the reduced evolution. The key observation is

that one has a spectral concentration in the set $\triangle_\varepsilon \equiv \triangle_\varepsilon(b) := (\epsilon_j - \varepsilon(b), \epsilon_j + \varepsilon(b))$ with a properly chosen $\varepsilon(b)$; we denote its complement as $\bar{\triangle}_\varepsilon := \sigma_{\mathrm{ac}}(H_{\alpha,\beta}) \setminus \triangle_\varepsilon$.

Lemma 7.2. *Suppose that $\varepsilon(b) \to 0$ and $\varepsilon(b)^{-1} b \to 0$ holds as $b \to 0$, then we have*

$$\|P_j \int_{\bar{\triangle}_\varepsilon} U_t(\lambda) \mathrm{d}E(\lambda)\psi_j\| \to 0\,.$$

Proof. Given an arbitrary Borel set $\triangle \subset \sigma_{\mathrm{ac}}(H_{\alpha,\beta})$ and a projection P we have the following simple inequality,

$$\|P \int_{\triangle} U_t(\lambda) \mathrm{d}E(\lambda) f\| \le \|E(\triangle) f\|\,, \tag{7.2}$$

and another straightforward application of the spectral theorem gives

$$\|(H_{\alpha,\beta} - \epsilon_j) f\|^2 \ge \int_{\bar{\triangle}_\varepsilon} |\lambda - \epsilon_j|^2 (\mathrm{d}E(\lambda) f, f) \ge \varepsilon(b)^2 \|E(\bar{\triangle}_\varepsilon) f\|^2 \tag{7.3}$$

for any $f \in D(H_{\alpha,\beta})$. To make use of the last inequality we need a suitable function from the domain of $H_{\alpha,\beta}$. It is clear that one cannot use ψ_j directly because it does not satisfy the appropriate boundary conditions at the line Σ, thus we take instead its modification $f_b = \psi_j + \phi_b$, where $\phi_b \in L^2(\mathbb{R}^2)$ vanishes on $\Pi \cup \Sigma$ and satisfies the following assumptions:

(a1) $\Xi_\Sigma(\phi_b) = -\alpha \Omega_\Sigma(\psi_j)$

(a2) $\|\phi_b\| = \mathcal{O}(b)$ and $\|\Delta \phi_b\| = \mathcal{O}(b)$.

In view of (3.2) the first condition guarantees that $f_b \in D(H_{\alpha,\beta})$, while the second one expresses "smallness" of the modification. It is not difficult to construct such a family. For instance, one can take for ϕ_b a family of C^2 functions with supports in a strip neighbourhood of Σ of width d_Σ assuming that ϕ_b behaves in the vicinity of Σ as $\frac{1}{2}\alpha \Omega_\Sigma(\psi_j)(x_1)|x_2|$. Since $|\Omega_\Sigma(\psi_j)| \le Cb$, where C is positive constant we can choose $d_\Sigma = \mathcal{O}(b)$. Using (a1) and $(\tilde{H}_\beta - \epsilon_j)\psi_j = 0$ we get

$$(H_{\alpha,\beta} - \epsilon_j) f_b = -\Delta \phi_b - \epsilon_j \phi_b\,,$$

so the condition (a2) gives

$$\|(H_{\alpha,\beta} - \epsilon_j) f_b\| = \mathcal{O}(b)\,.$$

This relation together with (7.3) yields $\|E(\bar{\triangle}_\varepsilon) f_b\| = \mathcal{O}(b)\varepsilon(b)^{-1}$. Combining it further with (7.2) and using the inequality

$$\|E(\bar{\triangle}_\varepsilon)\psi_j\| \le \|\phi_b\| + \|E(\bar{\triangle}_\varepsilon) f_b\|$$

and the condition (a2) we get the sought result. □

The next step is to show that the main contribution to the reduced evolution of the unstable state comes from the interval $\triangle_\varepsilon$.

Lemma 7.3. *Under the assumptions of Lemma 7.2 we have*

$$\|P_j \int_{\triangle_\varepsilon} U_t(\lambda) \mathrm{d}E(\lambda)\psi_j - U_t(z_j)\psi_j\| \to 0$$

for any fixed $t > 0$ in the limit $b \to 0$.

Proof. Let $R^{\mathrm{II}}_{\alpha,\beta}$ stand for the second-sheet continuation of the resolvent of $H_{\alpha,\beta}$. Using the results of Sec. 4 we can write it for a fixed j as

$$R^{\mathrm{II}}_{\alpha,\beta}(z) = \sum_{k=1}^{m} \frac{B_b^{(k)}}{z-z_k} + A_b(z)\,, \tag{7.4}$$

where $B_b^{(k)}$ is a one-parameter family of rank-one operators and $A_b(\cdot)$ is a family of analytic operator-valued functions to be specified later. Mimicking now the argument of [6, Sec. 3.1] which relies on Stone's formula and Radon-Nikodým theorem we find that the spectral-measure derivative acts at the vector ψ_j as

$$\frac{\mathrm{d}E(\lambda)}{\mathrm{d}\lambda}\,\psi_j = \left[\frac{1}{2\pi i}\sum_{k=1}^{m}\left(\frac{(B_b^{(k)})^*}{\lambda-\bar{z_k}} - \frac{B_b^{(k)}}{\lambda - z_k}\right) + \frac{1}{\pi}\,\mathrm{Im}\,A_b(\lambda)\right]\psi_j\,. \tag{7.5}$$

This makes it possible to estimate $P_j\int_{\triangle_\varepsilon} U_t(\lambda)\mathrm{d}E(\lambda)\psi_j$. Using the explicit form of $R^{\mathrm{II}}_{\alpha,\beta}$ derived in Sec. 4 one can check that $A_b(\cdot)$ can be bounded on a compact interval uniformly for b small enough, which means that the contribution to the integral from the last term in (7.5) tends to zero as $\varepsilon(b)\to 0$. The rest is dealt with by means of the residue theorem in the usual way: we can extend the integration to the whole real line and perform it by means of the integral over a closed contour consisting of a real axis segment and a semicircle in the lower halfplane, using the fact that the contribution from the latter vanishes when the semicircle radius tends to infinity. It is clear that only the m poles in (7.5) contained in the lower halfplane contribute, the k-th one giving $U_t(z_k)P_jB_b^{(k)}\psi_j$; an argument similar to Lemma 7.2 shows that the integral over $\mathbb{R}\setminus\triangle_\varepsilon$ vanishes as $b\to 0$, and likewise, the integral over semicircle vanishes in the limit of infinite radius.

Furthermore, since P_j is one-dimensional we have $P_jB_b^{(k)}\psi_j = c_b^{(k)}\psi_j$ where $b\mapsto c_b^{(k)}$ are continuous complex functions, well defined for b small enough. Hence the above discussion allows us to conclude that

$$\|P_j\mathrm{e}^{-iH_{\alpha,\beta}t}\psi_j - \sum_{k=1}^{m} c_b^{(k)}\mathrm{e}^{-iz_kt}\psi_j\| \to 0 \quad \text{as} \quad b\to 0\,. \tag{7.6}$$

Our next task is show that for $k\neq j$ we have $c_b^{(k)}\to 0$ as $b\to 0$ and $c_b^{(j)}\to 1$ at the same time. To this aim it suffices to check that $B_b^{(k)}$ converges to P_k for $b\to 0$. First we observe that the terms involved in the resolvent $R_{\alpha,\beta}$ derived in Theorem 3.1 satisfy the following relations

$$D(z)\to\Gamma_{11}(z)\,,\quad \mathbf{R}_{\alpha;1L}(z)\to\mathbf{R}_{1L}(z) \quad \text{as} \quad b\to 0$$

in the operator-norm sense; the limits are uniform on any compact subset of the upper halfplane as well as for the analytical continuation of $R_{\alpha,\beta}$. Consequently, the second component of the resolvent tends $\mathbf{R}_{L1}(z)[\Gamma_{11}(z)]^{-1}\mathbf{R}_{1L}(z)$ which obviously has a singular part equal to $\sum_{k=1}^{m}(z-\epsilon_k)^{-1}P_k$; this proves the claim. $\square$

Proof of Theorem 7.1. In view of (7.1) together with Lemmata 7.2, 7.3 it remains to demonstrate that the contribution from the discrete spectrum to (7.1) vanishes as $b \to 0$, i.e. that

$$\left| \sum_{k=1}^{m} U_t(\epsilon_{\alpha\beta,k}) |(\psi_j, \psi_{\alpha\beta,k})|^2 \right| \to 0\,. \tag{7.7}$$

This is a direct consequence of the following relation,

$$0 = (H_{\alpha,\beta}\psi_{\alpha\beta,k}, f_b) - (\psi_{\alpha\beta,k}, H_{\alpha,\beta} f_b) = (\epsilon_{\alpha\beta,k} - \epsilon_j)(\psi_{\alpha\beta,k}, f_b) + \mathcal{O}(b)\,,$$

where $k = 1, \dots, l$, and f_b is the function introduced in the proof of Lemma 7.2. In combination with (4.1) we get $|(\psi_j, \psi_{\alpha\beta,k})| = \mathcal{O}(b)$ which in turn implies (7.7).

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Acknowledgment

The research was partially supported by the ASCR and its Grant Agency within the projects IRP AV0Z10480505 and A100480501 and by the Polish Ministry of Scientific Research and Information Technology under the (solicited) grant no PBZ-Min-008/PO3/2003. Two of the authors (P.E. and S.K.) are grateful to the organizing committee of OTAMP2004 for supporting their participation in the conference, as well as for the hospitality at the University of Kanazawa, where a substantial part of this work was done.

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